通信工学概論
Introduction to Communication Engineering

第2回講義資料
Lecture notes 2

アナログ変調
Analog modulation

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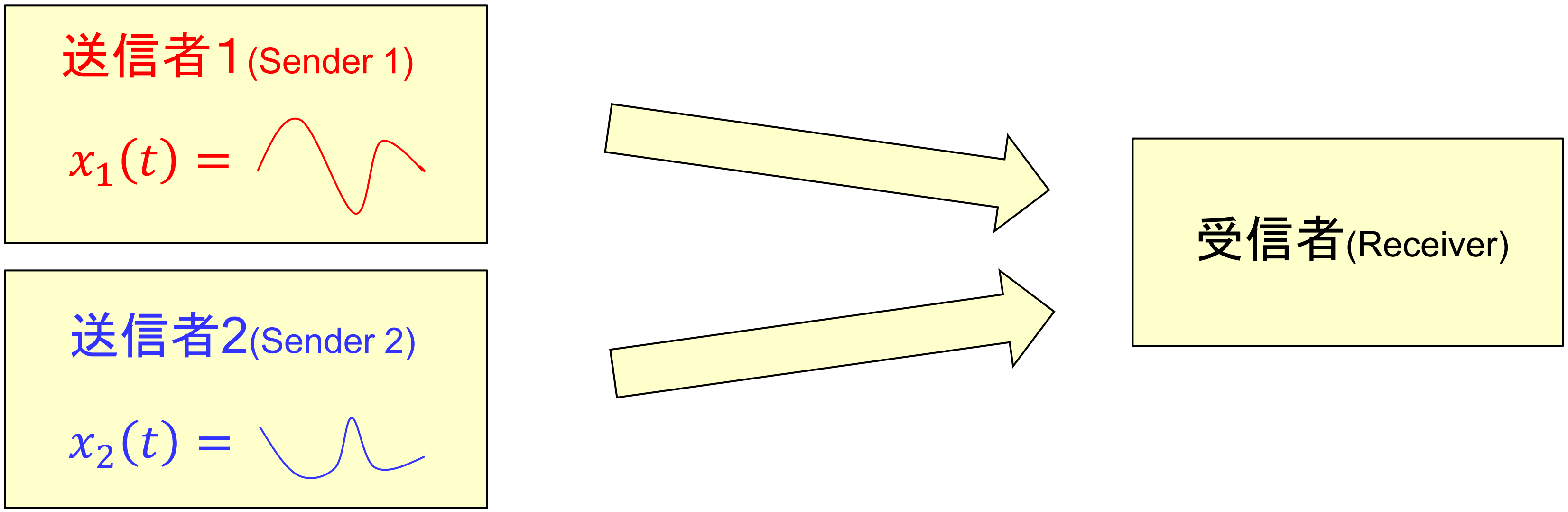
アナログ変調の目的 (Purpose of analog modulation)

二つの連続時間アナログ信号$x_1(t)$と$x_2(t)$を干渉なしで伝送したい。

Transmit two continuous-time analog signals $x_1(t)$ and $x_2(t)$ without interference.



二人の送信者が、同一通信路を使って同時伝送すると、干渉が生じる。

Interference occurs when two signals are sent over the same channel at the same time.

信号間の干渉を防ぐ目的で、変調を行う。

Modulation is used to circumvent the interference between the signals.

複素数の復習 (Review of complex numbers)

虚数単位j(Imaginary unit) $j^2 = -1$

複素数(Complex number) xとyを実数として、$z = x + jy$を複素数と呼ぶ。

$z = x + jy$ is called a complex number for real numbers x and y.

演算規則(Arithmetic operations)

二つの複素数$z_1 = x_1 + jy_1$と$z_2 = x_2 + jy_2$に対して、

For two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$,

$$z_1 \pm z_2 = (x_1 \pm x_2) + j(y_1 \pm y_2)$$

$$z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2) = x_1 x_2 + jy_1 x_2 + jx_1 y_2 + j^2 y_1 y_2$$

$$= x_1 x_2 - y_1 y_2 + j(x_1 y_2 + x_2 y_1) \quad \because j^2 = -1$$

複素共役(Complex conjugate) $\bar{z} = \overline{x + jy} = x - jy$

絶対値(Absolute value) $|z| = (z\bar{z})^{1/2} = \sqrt{(x + jy)(x - jy)} = \sqrt{x^2 + y^2}$

割り算と有理化(Division and rationalization)

二つの複素数$z_1 = x_1 + jy_1$と$z_2 = x_2 + jy_2$に対して、

For two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$,

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \frac{x_1x_2 + y_1y_2 + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

注意1(Remark 1)

$z_2 \neq 0$ならば、割り算は定義できる。

The division is well defined for $z_2 \neq 0$.

注意2(Remark 2)

分母の複素共役を分母と分子にかけて、分母からjを取り除く操作を有理化と呼ぶ。(赤色の等号参照)

Rationalization means operations to eliminate j in the denominator via the multiplication of both denominator and numerator by the complex conjugate of the denominator (See the red equality).

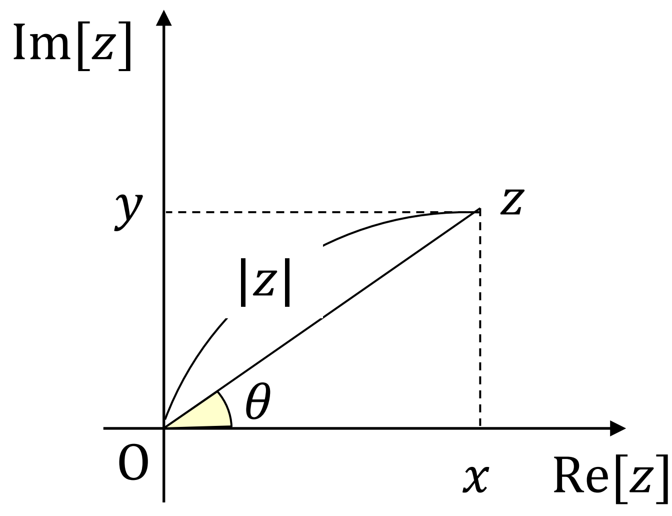
オイラーの公式と複素平面(Euler's formula and complex plane)

オイラーの公式(Euler's formula)

複素数$z=x+jy$に対して、(For a complex number $z=x+jy$,)

$$z=|z|(\cos\theta+j\sin\theta)\equiv|z|e^{j\theta},\qquad \tan\theta=\frac{y}{x}$$

複素平面(Complex plane)



zの実部xと虚部yをそれぞれ$\mathrm{Re}[z]$と$\mathrm{Im}[z]$と書く。

The real and imaginary parts of z are denoted by $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$, respectively.

三角関数の別表現(Another representation of Trigonometric functions)

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{j2}.$$

証明(Proof)

$$e^{j\theta} = \cos\theta + j\sin\theta\,,$$

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta\,.$$

上記の両辺を足す。(Add both sides above.)

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta\,. \qquad \therefore \quad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

前者の式から後者の式を引く。

Subtract the latter equation from the former equation.

$$e^{j\theta} - e^{-j\theta} = j2\sin\theta\,. \qquad \therefore \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{j2}. \qquad \blacksquare$$

指数法則(Exponents rules)

$z_1 = |z_1|(\cos\theta_1 + j\sin\theta_1)$と$z_2 = |z_2|(\cos\theta_2 + j\sin\theta_2)$に対して、

For $z_1 = |z_1|(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + j\sin\theta_2)$,

$$z_1 z_2 = |z_1||z_2|e^{j\theta_1}e^{j\theta_2} = |z_1 z_2|e^{j(\theta_1+\theta_2)}$$

証明(Proof) $|z_1||z_2| = |z_1 z_2|$は自明なので、$|z_1| = |z_2| = 1$を仮定する。

Assume $|z_1| = |z_2| = 1$ since $|z_1||z_2| = |z_1 z_2|$ is trivial.

複素指数関数を以下で定義する。(Define the complex exponential function as follows:)

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

複素指数関数は指数法則$\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$を満たすので、

The complex exponential function satisfies the exponent rule $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$. Thus,

$$\exp(j\theta_1)\exp(j\theta_2) = \exp\big(j(\theta_1 + \theta_2)\big).$$

$\exp(j\theta) = e^{j\theta}$を確認すればよい。(It is sufficient to confirm $\exp(j\theta) = e^{j\theta}$.)

証明(Proof)

$(j\theta)^{2k}=(-1)^k\theta^{2k}$と$(j\theta)^{2k+1}=j(-1)^k\theta^{2k+1}$から、

From $(j\theta)^{2k}=(-1)^k\theta^{2k}$ and $(j\theta)^{2k+1}=j(-1)^k\theta^{2k+1}$,

$$\exp(j\theta)=\sum_{n=0}^{\infty}\frac{(j\theta)^n}{n!}=\underbrace{\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!}}_{=C(\theta)}+j\underbrace{\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}}_{=S(\theta)}.$$

$C(\theta)=\cos\theta$と$S(\theta)=\sin\theta$を確かめる。(Confirm $C(\theta)=\cos\theta$ and $S(\theta)=\sin\theta$.)

$$S'(\theta)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\theta^{2k+1}\right)'}{(2k+1)!}=\sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!}=C(\theta),\qquad S(0)=0$$

$$C'(\theta)=\sum_{k=1}^{\infty}\frac{(-1)^k\theta^{2k-1}}{(2k-1)!}=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\theta^{2k+1}}{(2k+1)!}=-S(\theta),\qquad C(0)=1$$

$\sin'\theta=\cos\theta$、$\sin 0=0$、$\cos'\theta=-\sin\theta$、$\cos 0=1$なので、微分方程式の解の一意性から、$C(\theta)=\cos\theta$と$S(\theta)=\sin\theta$を得る。

Since $\sin'\theta=\cos\theta$, $\sin 0=0$, $\cos'\theta=-\sin\theta$, and $\cos 0=1$ hold, the uniqueness of the solution to differential equations implies $C(\theta)=\cos\theta$ and $S(\theta)=\sin\theta$. ■

三角関数の公式(Trigonometric identities)

ピタゴラスの定理(Pythagorean theorem)

$$\cos^2\theta + \sin^2\theta = 1$$

$\because$ $f(\theta) = \cos^2\theta + \sin^2\theta$とおく。(Let $f(\theta) = \cos^2\theta + \sin^2\theta$.)

$$f'(\theta) = 2\cos\theta\,(-\sin\theta) + 2\sin\theta\cos\theta = 0, \qquad f(0) = 1. \qquad \blacksquare$$

和積公式(Sum-to-product formulas)

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2\,,$$

$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2\,.$$

$\because$ $e^{j(\theta_1 \pm \theta_2)} = e^{j\theta_1}e^{\pm j\theta_2}$の両辺を評価せよ。(Evaluate both sides on $e^{j(\theta_1 \pm \theta_2)} = e^{j\theta_1}e^{\pm j\theta_2}$.)

左辺(Left-hand side) $e^{j(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)\,.$

右辺(Right-hand side)

$$e^{j\theta_1}e^{\pm j\theta_2} = (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 \pm j\sin\theta_2)$$

$$= \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 + j(\sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2). \quad \blacksquare$$

三角関数の公式(Trigonometric identities)

積和公式

Product-to-sum formulas

$$\cos\theta_1\cos\theta_2 = \frac{\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)}{2},$$

$$\sin\theta_1\sin\theta_2 = \frac{-\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2)}{2},$$

$$\sin\theta_1\cos\theta_2 = \frac{\sin(\theta_1+\theta_2)+\sin(\theta_1-\theta_2)}{2}.$$

∵ 和積公式より、

From the sum-to-product formulas,

$$\cos(\theta_1+\theta_2)+\cos(\theta_1-\theta_2) = 2\cos\theta_1\cos\theta_2 .$$

$$\cos(\theta_1+\theta_2)-\cos(\theta_1-\theta_2) = -2\sin\theta_1\sin\theta_2 .$$

$$\sin(\theta_1+\theta_2)+\sin(\theta_1-\theta_2) = 2\sin\theta_1\cos\theta_2 . \quad \blacksquare$$

2倍角の公式

Double-angle formulas

$$\cos(2\theta) = 2\cos^2\theta - 1, \qquad \sin(2\theta) = 2\sin\theta\cos\theta .$$

∵ 和積公式で$\theta_1 = \theta_2 = \theta$とすると、(Let $\theta_1 = \theta_2 = \theta$ in the sum-to-product formulas.)

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1.$$

最後の等号の導出で、$\cos^2\theta + \sin^2\theta = 1$を使った。

In the derivation of the last equality, we have used $\cos^2\theta + \sin^2\theta = 1$.

$$\sin(2\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta . \quad \blacksquare$$

三角関数の直交性 (Orthogonality between trigonometric functions)

直交性(Orthogonality)

区間$[a,b]$上の関数fとgに対して、以下が成立するとき、fとgは直交すると言う。

Two functions f and g on the interval $[a,b]$ are said to be orthogonal if the following is satisfied:

$$\langle f,g\rangle \equiv \int_a^b f(t)g(t)dt = 0.$$

補題2.1(Lemma 2.1)

非負の整数kとk'に対して、以下の直交性が成立する。

For non-negative integers k and k', the following orthogonality holds:

$$\int_{-T/2}^{T/2} \cos\left(\frac{2k\pi t}{T}\right)\cos\left(\frac{2k'\pi t}{T}\right)dt = \begin{cases} T & \text{for } k = k' = 0, \\ T/2 & \text{for } k = k' \neq 0, \\ 0 & \text{for } k \neq k', \end{cases}$$

$$\int_{-T/2}^{T/2} \sin\left(\frac{2k\pi t}{T}\right)\sin\left(\frac{2k'\pi t}{T}\right)dt = \begin{cases} T/2 & \text{for } k = k' \neq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\int_{-T/2}^{T/2} \sin\left(\frac{2k\pi t}{T}\right)\cos\left(\frac{2k'\pi t}{T}\right)dt = 0.$$

補題2.1の証明 (Proof of Lemma 2.1)

表記の簡単化のため、$T' = T/2$とおき、T'をTと書く。

For notational simplicity, we let $T' = T/2$ and rewrite T' as T.

$k = 0$の場合、

For $k = 0$,

$$\int_{-T}^{T} \sin 0 \sin\left(\frac{k'\pi t}{T}\right) dt = 0, \qquad \int_{-T}^{T} \sin 0 \cos\left(\frac{k'\pi t}{T}\right) dt = 0.$$

さらに$k = k' = 0$の場合、

Furthermore, for $k = k' = 0$

$$\int_{-T}^{T} \cos^2 0\, dt = [t]_{-T}^{T} = 2T.$$

$k = k' \neq 0$の場合、2倍角の公式を使って、

For $k = k' \neq 0$, using the double-angle formula yields

$$\int_{-T}^{T} \cos^2\left(\frac{k\pi t}{T}\right) dt = \int_{-T}^{T} \frac{\cos(2k\pi t/T) + 1}{2} dt = \left[\frac{T}{4k\pi} \sin\left(\frac{2k\pi t}{T}\right) + \frac{t}{2}\right]_{-T}^{T} = T.$$

$$\int_{-T}^{T} \sin^2\left(\frac{k\pi t}{T}\right) dt = \int_{-T}^{T} \left\{1 - \cos^2\left(\frac{k\pi t}{T}\right)\right\} dt = 2T - T = T.$$

$$\int_{-T}^{T} \sin\left(\frac{k\pi t}{T}\right) \cos\left(\frac{k\pi t}{T}\right) dt = \frac{1}{2}\int_{-T}^{T} \sin\left(\frac{2k\pi t}{T}\right) dt = \left[-\frac{T}{4k\pi} \cos\left(\frac{2k\pi t}{T}\right)\right]_{-T}^{T} = 0.$$

補題2.1の証明 (Proof of Lemma 2.1)

$k \neq k'$の場合、積和公式を使って、(For $k \neq k'$, using the product-to-sum formula yields)

$$\int_{-T}^{T} \cos\left(\frac{k\pi t}{T}\right)\cos\left(\frac{k'\pi t}{T}\right)dt = \frac{1}{2}\int_{-T}^{T}\left\{\cos\left(\frac{(k+k')\pi t}{T}\right) + \cos\left(\frac{(k-k')\pi t}{T}\right)\right\}dt$$

$$= \frac{1}{2}\left[\frac{T}{(k+k')\pi}\sin\left(\frac{(k+k')\pi t}{T}\right)\right]_{-T}^{T} + \frac{1}{2}\left[\frac{T}{(k-k')\pi}\sin\left(\frac{(k-k')\pi t}{T}\right)\right]_{-T}^{T} = 0.$$

$$\int_{-T}^{T} \sin\left(\frac{k\pi t}{T}\right)\sin\left(\frac{k'\pi t}{T}\right)dt = \frac{1}{2}\int_{T}^{T}\left\{\cos\left(\frac{(k-k')\pi t}{T}\right) - \cos\left(\frac{(k+k')\pi t}{T}\right)\right\}dt$$

$$= \frac{1}{2}\left[\frac{T}{(k-k')\pi}\sin\left(\frac{(k-k')\pi t}{T}\right)\right]_{-T}^{T} - \frac{1}{2}\left[\frac{T}{(k+k')\pi}\sin\left(\frac{(k+k')\pi t}{T}\right)\right]_{-T}^{T} = 0.$$

$$\int_{-T}^{T} \sin\left(\frac{k\pi t}{T}\right)\cos\left(\frac{k'\pi t}{T}\right)dt = \frac{1}{2}\int_{-T}^{T}\left\{\sin\left(\frac{(k+k')\pi t}{T}\right) + \sin\left(\frac{(k-k')\pi t}{T}\right)\right\}dt$$

$$= -\frac{1}{2}\left[\frac{T}{(k+k')\pi}\cos\left(\frac{(k+k')\pi t}{T}\right)\right]_{-T}^{T} - \frac{1}{2}\left[\frac{T}{(k-k')\pi}\cos\left(\frac{(k-k')\pi t}{T}\right)\right]_{-T}^{T} = 0. \quad \blacksquare$$

一般化フーリエ級数(Generalized Fourier series)

直交基底関数(Orthogonal basis function)

区間$[a,b]$上の任意の関数gが、直交関数$\{\phi_k(t)\}_{k=1}^{\infty}$の線形結合で表現できるとき、$\{\phi_k(t)\}_{k=1}^{\infty}$は直交基底関数と呼ばれる。

Orthogonal functions $\{\phi_k(t)\}_{k=1}^{\infty}$ are said to be orthogonal basis functions if any function g on the interval $[a,b]$ is represented as a linear combination of the functions $\{\phi_k(t)\}_{k=1}^{\infty}$.

$$g(t) = \sum_{k=1}^{\infty} a_k \phi_k(t).$$

一般化フーリエ係数(Generalized Fourier coefficients)

$$a_k = \frac{\langle g, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle}.$$

$$\because \quad \langle g, \phi_k \rangle = \int_a^b g(t)\phi_k(t)dt = \int_a^b \sum_{k'=1}^{\infty} a_{k'}\phi_{k'}(t)\phi_k(t)\,dt$$

$$= \sum_{k'=1}^{\infty} a_{k'}\langle \phi_{k'}, \phi_k \rangle = a_k \langle \phi_k, \phi_k \rangle \quad \because \langle \phi_{k'}, \phi_k \rangle = 0 \text{ for } k \neq k'. \quad \blacksquare$$

フーリエ級数(Fourier series)

フーリエ基底(Fourier basis)

$\{1, \cos(2k\pi t/T), \sin(2k\pi t/T)\}_{k=1}^{\infty}$は、区間$[-T/2, T/2]$上の二乗可積分関数$g$の空間に対する直交基底関数である。

$\{1, \cos(2k\pi t/T), \sin(2k\pi t/T)\}_{k=1}^{\infty}$ are orthogonal basis functions for the space of square-integrable functions g on the interval $[-T/2, T/2]$.

$$g(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos\left(\frac{2k\pi t}{T}\right) + b_k \sin\left(\frac{2k\pi t}{T}\right) \right\}.$$

フーリエ係数(Fourier coefficients)

$$\frac{a_0}{2} = \frac{\langle g, 1 \rangle}{\langle 1, 1 \rangle} = \frac{1}{T} \int_{-T/2}^{T/2} g(t)\, dt,$$

$$a_k = \frac{\langle g, \cos(2k\pi t/T) \rangle}{\langle \cos(2k\pi t/T), \cos(2k\pi t/T) \rangle} = \frac{2}{T} \int_{-T/2}^{T/2} g(t) \cos\left(\frac{2k\pi t}{T}\right) dt,$$

$$b_k = \frac{\langle g, \sin(2k\pi t/T) \rangle}{\langle \sin(2k\pi t/T), \sin(2k\pi t/T) \rangle} = \frac{2}{T} \int_{-T/2}^{T/2} g(t) \sin\left(\frac{2k\pi t}{T}\right) dt.$$

∵ 補題2.1 (Lemma 2.1)

アナログ振幅変調(Analog amplitude modulation)

仮定(Assumption)

搬送波$\cos(2\pi f_i t)$の周期$T_i = 1/f_i$は、送信信号$x_1(t)$と$x_2(t)$の時変動のスケールよりも、十分に短いと仮定する。

Suppose that the period $T_i = 1/f_i$ of a carrier wave $\cos(2\pi f_i t)$ is much smaller than the time scale of the transmitted signals $x_1(t)$ and $x_2(t)$.

振幅変調(Amplitude modulation: AM)

送信者iは、$g_i(t) = x_i(t)\cos(2\pi f_i t)$を送る。(Sender i transmits $g_i(t) = x_i(t)\cos(2\pi f_i t)$.)

受信信号(Received signal)

$$y(t) = x_1(t)\cos(2\pi f_1 t) + x_2(t)\cos(2\pi f_2 t)\,.$$

受信信号$y(t)$から、$x_1(t)$に関する情報だけを取り出したい。

Extract the information on $x_1(t)$ from the received signal $y(t)$.

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復調(Demodulation)

$f_1 = kf$と$f_2 = k'f$が基本周波数$f = 1/T$の整数倍$(k \neq k')$かつ$f \gg 1$の場合、

When $f_1 = kf$ and $f_2 = k'f$ are integer multiples $(k \neq k')$ of a fundamental frequency $f = 1/T \gg 1$,

同期検波

Coherent reception

$$\int_{-T/2}^{T/2} y(t)\cos\left(\frac{2k\pi t}{T}\right)dt = \int_{-T/2}^{T/2} x_1(t)\cos^2\left(\frac{2k\pi t}{T}\right)dt + \int_{-T/2}^{T/2} x_2(t)\cos\left(\frac{2k'\pi t}{T}\right)\cos\left(\frac{2k\pi t}{T}\right)dt$$

$x_i(t)$は、$-T/2 \leq t \leq T/2$の場合に定数と近似すると、

Regard $x_i(t)$ as a constant for $-T/2 \leq t \leq T/2$ approximately.

所望信号

Desired signal

$$\int_{-T/2}^{T/2} x_1(t)\cos^2\left(\frac{2k\pi t}{T}\right)dt \approx x_1(t)\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos^2\left(\frac{2k\pi t}{T}\right)dt = \frac{T}{2}x_1(t),$$

干渉信号

Interference

$$\int_{-T/2}^{T/2} x_2(t)\cos\left(\frac{2k'\pi t}{T}\right)\cos\left(\frac{2k\pi t}{T}\right)dt$$

$$\approx x_2(t)\int_{-T/2}^{T/2} \cos\left(\frac{2k'\pi x}{T}\right)\cos\left(\frac{2k\pi x}{T}\right)dt = 0.$$

∵ 補題2.1

Lemma 2.1

演習(Exercise)

アナログ信号$\omega(t)$を搬送波$\cos(2\pi f_c t)$の位相に変調して、送信する。

Transmit an analog signal $\omega(t)$ via modulation of $\omega(t)$ for the phase of the carrier wave $\cos(2\pi f_c t)$.

$$x(t) = \cos(2\pi f_c t + \omega(t)) .$$

1. 以下の$y(t)$を計算せよ。ただし、$f_c \gg 1$を仮定して、$t \le \tau \le t + f_c^{-1}$に対して$\omega(\tau) \approx \omega(t)$と近似せよ。

   Compute the following $y(t)$ by using the approximation $\omega(\tau) = \omega(t)$ for $t \le \tau \le t + f_c^{-1}$ under the assumption $f_c \gg 1$.

$$y(t) = 2f_c \int_t^{t+f_c^{-1}} x(\tau) \cos(2\pi f_c \tau)\, d\tau .$$

2. 信号$y(t)$から近似的に$\omega(t)$を一意に復調できるために、$\omega(t)$が取りうる値の条件を答えよ。

   Answer conditions for the image of $\omega(t)$ to demodulate $\omega(t)$ from $y(t)$ approximately and uniquely.