信号解析論
Signal Processing
第11回講義資料
Lecture notes 11
離散時間線形時不変システム2
Discrete-time and linear time-invariant systems 2

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先週の復習(Review of the last week)

離散時間線形時不変システム(Discrete-time linear time-invariant systems)

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

入力信号(Input signals)

$\{x[n]\}_{n=-\infty}^{\infty}$

インパルス応答

Impulse response

$\{h[n]\}_{n=-\infty}^{\infty}$

出力信号(Output signals)

$\{y[n]\}_{n=-\infty}^{\infty}$

伝達関数(Transfer function)

$\{x[n]\}_{n=-\infty}^{\infty}$、$\{y[n]\}_{n=-\infty}^{\infty}$、$\{h[n]\}_{n=-\infty}^{\infty}$のZ変換を$X(z)$、$Y(z)$、$H(z)$とする。

Let $X(z)$, $Y(z)$, and $H(z)$ denote the Z-transforms of $\{x[n]\}_{n=-\infty}^{\infty}$, $\{y[n]\}_{n=-\infty}^{\infty}$, and $\{h[n]\}_{n=-\infty}^{\infty}$, respectively.

$$Y(z) = H(z)X(z) \qquad \because \text{定理9.2(Theorem 9.2)}$$

インパルス応答のZ変換$H(z)$を伝達関数と呼ぶ。

The Z-transform $H(z)$ of the impulse response is called transfer function.

周波数特性(Frequency response)

正弦波入力(Sinusoidal input)　$x[n] = e^{2\pi jfn}$

出力(Output)

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m] = e^{2\pi jfn} \sum_{m=-\infty}^{\infty} h[m]e^{-2\pi jfm}$$

$$= e^{2\pi jfn} H\left(e^{2\pi jf}\right) = H\left(e^{2\pi jf}\right)x[n]$$

正弦波入力に$H\left(e^{2\pi jf}\right)$が乗算される。

The sinusoidal input is multiplied by $H\left(e^{2\pi jf}\right)$.

周波数特性(Frequency response)

$H\left(e^{2\pi jf}\right)$を周波数特性と呼ぶ。($H\left(e^{2\pi jf}\right)$ is called frequency response.)

特に、$H\left(e^{2\pi jf}\right) = A(f)\exp\{2\pi jF(f)\}$と表現したとき、$A(f) \geq 0$と$2\pi F(f) \in \mathbb{R}$をそれぞれ周波数**振幅特性**と周波数**位相特性**と呼ぶ。

In particular, $A(f) \geq 0$ and $2\pi F(f) \in \mathbb{R}$ are called amplitude and phase responses, respectively, for $H\left(e^{2\pi jf}\right) = A(f)\exp\{2\pi jF(f)\}$.

周波数特性の対称性(Symmetries of the frequency response)

性質11.1 $\{h[n] \in \mathbb{R}\}$を実インパルス応答とする。

Property 11.1 Suppose that an impulse response $\{h[n] \in \mathbb{R}\}$ is real.

$$H\left(e^{-2\pi jf}\right) = \overline{H\left(e^{2\pi jf}\right)}$$

$$\because \quad H\left(e^{-2\pi jf}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{2\pi jfn} = \sum_{n=-\infty}^{\infty} \overline{h[n]}\,\overline{e^{-2\pi jfn}} = \overline{H\left(e^{2\pi jf}\right)} \qquad \blacksquare$$

性質11.2 $\{h[n] \in \mathbb{R}\}$を実インパルス応答とする。

Property 11.2 Suppose that an impulse response $\{h[n] \in \mathbb{R}\}$ is real.

$$A(-f) = A(f)$$

$$\because A(-f) = \left|H\left(e^{-2\pi jf}\right)\right| = \left|\overline{H\left(e^{2\pi jf}\right)}\right| = \left|H\left(e^{2\pi jf}\right)\right| = A(f) \qquad \blacksquare$$

性質11.3 $\{h[n] \in \mathbb{R}\}$を実インパルス応答とする。

Property 11.3 Suppose that an impulse response $\{h[n] \in \mathbb{R}\}$ is real.

$$F(-f) = -F(f) + k \text{ for all } k \in \mathbb{Z}$$

$$\because H\left(e^{-2\pi jf}\right) = \overline{H\left(e^{2\pi jf}\right)},\ A(-f) = A(f) \Longrightarrow e^{2\pi jF(-f)} = e^{-2\pi jF(f)} \qquad \blacksquare$$

群遅延と線形位相特性(Group delay and linear phase response)

定義11.1(Definition 11.1)

$H(e^{2\pi jf}) \neq 0$を満たすfに対して、$H(e^{2\pi jf}) = e^{2\pi jF(f)}|H(e^{2\pi jf})|$と表現したとき、以下で定義される$\tau_{\mathrm{g}}(f)$を群遅延と呼ぶ。

Let $H(e^{2\pi jf}) = e^{2\pi jF(f)}|H(e^{2\pi jf})|$ denote a frequency response for f satisfying $H(e^{2\pi jf}) \neq 0$, the function $\tau_{\mathrm{g}}(f)$ is called group delay.

$$\tau_{\mathrm{g}}(f) = -\frac{d(2\pi F)}{d(2\pi f)} = -\frac{dF}{df}$$

※通常の群遅延は、fではなく、$\omega = 2\pi f$の関数として定義される。

The group delay is usually defined as a function of $\omega = 2\pi f$, rather than f.

定義11.2(Definition 11.2)

群遅延$\tau_{\mathrm{g}}(f)$が定数のとき、$H(e^{2\pi jf})$は線形位相特性を持つと言う。

We say that $H(e^{2\pi jf})$ has a linear phase response if the group delay $\tau_{\mathrm{g}}(f)$ is constant.

線形位相特性(Linear phase response)

$H(e^{2\pi jf}) \neq 0$を満たすfに対して、(For f satisfying $H(e^{2\pi jf}) \neq 0$,)

$$H(e^{2\pi jf}) = e^{2\pi j(af+b)}|H(e^{2\pi jf})|, \qquad a, b \in \mathbb{R}$$

※通常の線形位相特性は、$A(f)$を実関数として、$H(e^{2\pi jf}) = e^{2\pi j(af+b)}A(f)$として定義される。

The linear phase response is usually defined via $H(e^{2\pi jf}) = e^{2\pi j(af+b)}A(f)$ for a real function $A(f)$.

位相遅延(Phase delay)

定義11.3(Definition 11.3)

$H\left(e^{2\pi jf}\right) \neq 0$を満たす$f$に対して、$H\left(e^{2\pi jf}\right) = e^{2\pi jF(f)}\left|H\left(e^{2\pi jf}\right)\right|$と表現したとき、以下で定義される$\tau_{\mathrm{p}}(f)$を位相遅延と呼ぶ。

Let $H\left(e^{2\pi jf}\right) = e^{2\pi jF(f)}\left|H\left(e^{2\pi jf}\right)\right|$ denote a frequency response for f satisfying $H\left(e^{2\pi jf}\right) \neq 0$, the function $\tau_{\mathrm{p}}(f)$ is called phase delay.

$$\tau_{\mathrm{p}}(f) = -\frac{2\pi F}{2\pi f} = -\frac{F}{f}$$

例11.1 Example 11.1

以下の時間遅延フィルタの位相遅延を評価せよ。

Evaluate the phase delay of the following time-delay filter:

$$h[n] = \delta_{n,n_0}, \qquad y[n] = \sum_{m=-\infty}^{\infty} \delta_{m,n_0}\, x[n-m] = x[n-n_0]$$

周波数特性$H\left(e^{2\pi jf}\right)$を求めると、

Evaluate the frequency response $H\left(e^{2\pi jf}\right)$.

$$H\left(e^{2\pi jf}\right) = \sum_{n=-\infty}^{\infty} \delta_{n,n_0} e^{-2\pi jfn} = e^{-2\pi jfn_0}$$

$$\therefore F(f) = -fn_0, \qquad \tau_{\mathrm{p}}(f) = -\frac{F(f)}{f} = n_0$$

位相遅延は時間遅延n_0に等しい。(The phase delay is equal to the time delay n_0.)

群遅延の意義(Significance of group delay)

例11.2 振幅変調信号$x[n]=a[n]e^{2\pi j f_c n}$を以下のフィルタに入力する。位相遅延と群遅延を求めよ。

Example 11.2 Input an amplitude modulation signal $x[n]=a[n]e^{2\pi j f_c n}$ to the following filter:

$$h[n]=e^{2\pi j(f_c n-\tau_0)}\delta_{n,n_0},\quad y[n]=\sum_{m=-\infty}^{\infty}h[m]x[n-m]$$

$$y[n]=\sum_{m=-\infty}^{\infty}e^{2\pi j(f_c m-\tau_0)}\delta_{m,n_0}a[n-m]e^{2\pi j f_c(n-m)}$$

$$=e^{-2\pi j\tau_0}e^{2\pi j f_c n}a[n-n_0]$$

周波数特性$H\left(e^{2\pi jf}\right)$を求めると、(Evaluate the frequency response $H\left(e^{2\pi jf}\right)$.)

$$H\left(e^{2\pi jf}\right)=e^{-2\pi j\tau_0}\sum_{n=-\infty}^{\infty}\delta_{n,n_0}e^{-2\pi j(f-f_c)n}=e^{-2\pi j\{\tau_0+(f-f_c)n_0\}}$$

位相遅延(Phase delay)

$$\tau_p(f)=\frac{\tau_0+(f-f_c)n_0}{f}$$

群遅延(Group delay)

$$\tau_g(f)=n_0$$

包絡線波形$a[n]$の遅延

(Delay of the envelope $a[n]$)

FIRシステムの例(Example of an FIR system)

例11.3 以下のインパルス応答を持つシステムの周波数特性、位相遅延、群遅延を評価せよ。

Example 11.3 Evaluate the frequency response, phase delay, and group delay of a system with the following impulse response:

$$h[n] = \begin{cases} 2^{-1} & \text{for } n = 0, 1, \\ 0 & \text{otherwise} \end{cases}$$

周波数特性$H\left(e^{2\pi jf}\right)$を求めると、

Evaluate the frequency response $H\left(e^{2\pi jf}\right)$.

$$H(z) = \sum_{n=0}^{1} \frac{1}{2} z^{-n} = \frac{1}{2} \frac{1 - z^{-2}}{1 - z^{-1}}$$

$$H\left(e^{2\pi jf}\right) = \frac{1}{2} \frac{1 - e^{-4\pi jf}}{1 - e^{-2\pi jf}} = \frac{e^{-2\pi jf}\left(e^{2\pi jf} - e^{-2\pi jf}\right)}{2e^{-\pi jf}\left(e^{\pi jf} - e^{-\pi jf}\right)} = e^{-\pi jf} \frac{\sin 2\pi f}{2 \sin \pi f}$$

振幅特性(Amplitude response)

$$A(f) = \left| \frac{\sin 2\pi f}{2 \sin \pi f} \right|$$

位相特性(Phase response)

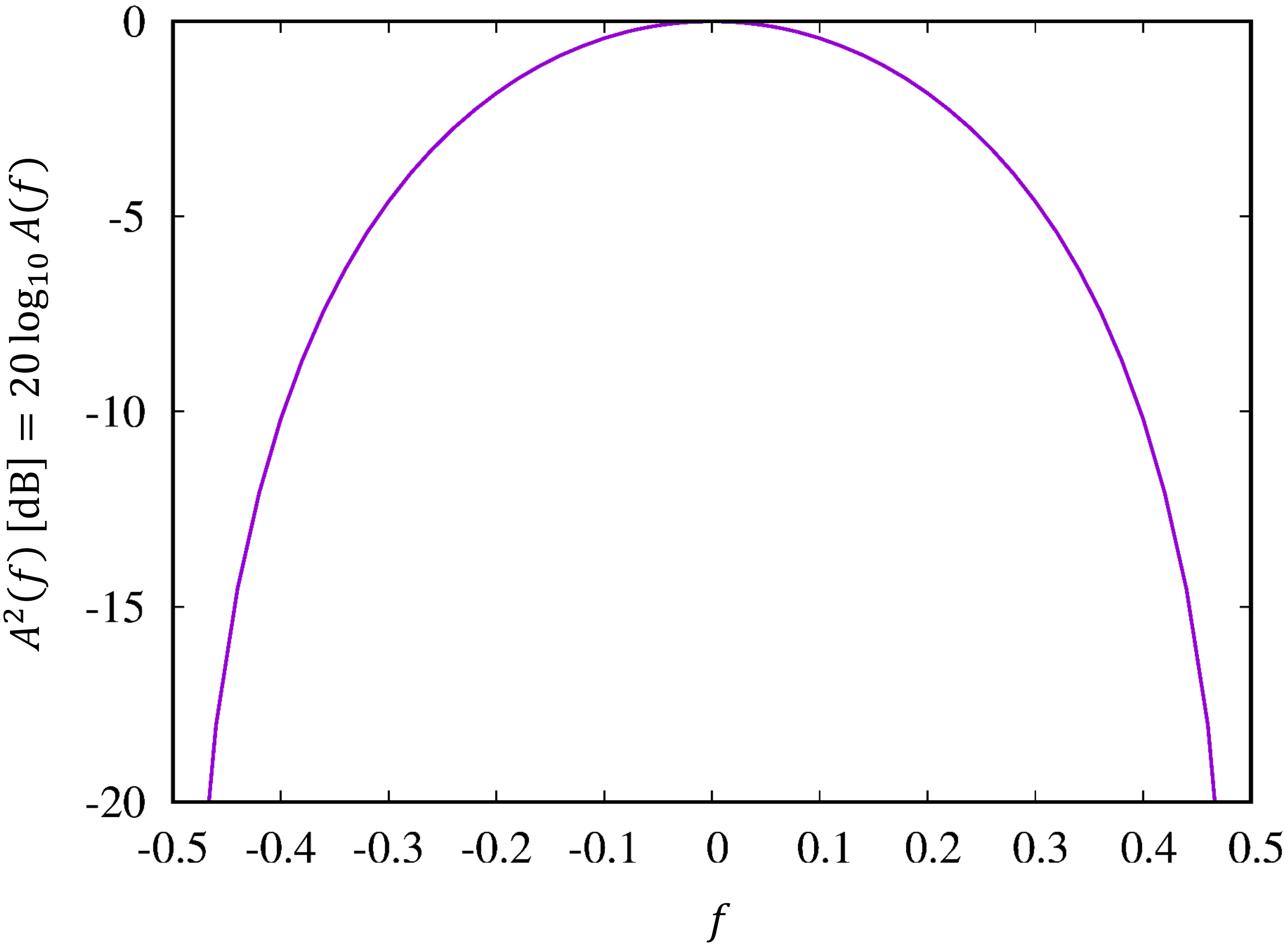
$$2\pi F(f) = -\pi f$$

位相遅延と群遅延

(Phase delay and group delay)

$$\tau_{\mathrm{p}}(f) = \tau_{\mathrm{g}}(f) = \frac{1}{2}$$

振幅特性(Amplitude response)



IIRシステムの例(Example of an IIR system)

例１１.４　以下のシステムの周波数特性を評価せよ。

Example 11.4　Evaluate the frequency response of the following system:

$$y[n] = x[n] + \frac{1}{2} y[n-1], \qquad x[n] = y[n] = 0 \text{ for } n < 0$$

性質9.4を使って両辺のZ変換を計算すると、

Use Property 9.4 to compute the Z-transforms of both sides.

$$Y(z) = X(z) + \frac{1}{2} z^{-1} Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{2 - z^{-1}} \qquad \therefore H\left(e^{2\pi jf}\right) = \frac{2}{2 - e^{-2\pi jf}} = \frac{2}{2 - \cos 2\pi f + j \sin 2\pi f}$$

振幅特性

Amplitude response

$$A(f) = \frac{2}{\sqrt{(2 - \cos 2\pi f)^2 + \sin^2 2\pi f}} = \frac{2}{\sqrt{5 - 4\cos 2\pi f}}$$

位相特性

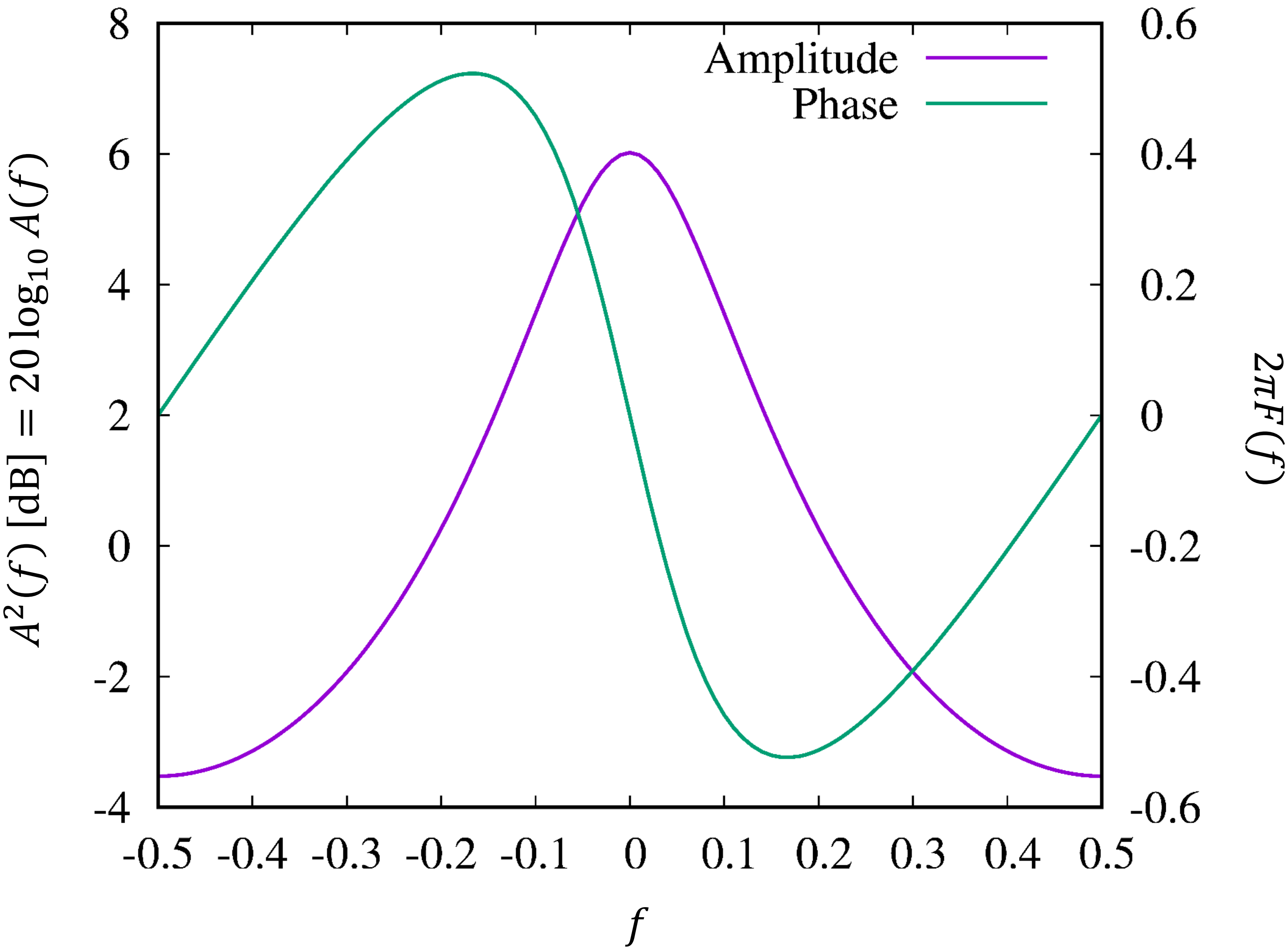
Phase response

$$2\pi F(f) = -\tan^{-1}\left(\frac{\sin 2\pi f}{2 - \cos 2\pi f}\right)$$

IIRシステムは線形位相特性を持たない。

The IIR system does not linear phase response.

振幅・位相特性(Amplitude and phase responses)



演習(Exercises)

次のIIRシステムの周波数特性を評価せよ。

Evaluate the frequency response of the following IIR system:

$$y[n] = x[n] - x[n-1] - y[n-1], \qquad x[n] = y[n] = 0 \text{ for } n < 0$$